Harmonic Oscillator—Revisited:

Coherent States

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December 2012

Introduction. Here I digress from work in progress—namely, a review of a paper by C. Y. She & H. Heffner[1], which was the first of several papers inspired by E. Arthurs & J. L. Kelly's "On the simultaneous measurement of a pair of conjugate observables" (BSTJ **44**, 725 (1965)); it is my intention to incorporate that material into an account of generalized quantum measurement theory on which I was at work two digressions ago—to sharpen my understanding of some elementary material that originated in Dirac's treatment of the theory of quantum oscillators, but which acquired new interest and was carried to a higher state of development when Roy Glauber[2] and others laid the foundations of quantum optics. My interest here is not in quantum optics but in the details of those mathematical "higher developments," as they relate oscillators and to the argument that lies at the heart of the She/Heffner paper.

I expect to draw my material primarily from Chapters 2 & 3 of Christopher Gerry and Peter Knight's *Introductory Quantum Optics* (2005) and Chapter 0 of my own *Advanced Quantum Topics* (2000).

Oscillator basics. The oscillator Hamiltonian reads

$$\mathsf{H} = \tfrac{1}{2m}(\mathsf{p}^2 + m^2\omega^2\mathsf{q}^2)$$

We set $m = 1$ and introduce non-hermitian operators

$$\left.\begin{aligned} \mathsf{a} \;\, &= \tfrac{1}{\sqrt{2\hbar\omega}}(\omega\mathsf{q} + i\mathsf{p}) \\ \mathsf{a}^+ &= \tfrac{1}{\sqrt{2\hbar\omega}}(\omega\mathsf{q} - i\mathsf{p}) \end{aligned}\right\} \tag{1}$$

giving

$$\begin{gathered} \mathsf{H} = \hbar\omega(\mathsf{a}^+\mathsf{a} + \tfrac{1}{2}\mathsf{I}) \\ [\mathsf{a}, \mathsf{a}^+] = \mathsf{I} \end{gathered} \tag{2}$$

Let the dimensionless hermitian "number operator" N—the name of which makes natural sense only in quantum-field-theoretic or quantum optical contexts

[1] "Simultaneous measurement of noncommuting observables," Phys. Rev. **152**, 1103 (1966).

[2] "Coherent and incoherent states of the radiation field," Phys. Rev. **131**, 2766 (1963).

where one is working either explicitly or implicitly in Foch space—be defined

$$\mathsf{N} = \mathsf{a}^+\mathsf{a} \tag{3}$$

It becomes more natural henceforth to speak of N than of $\mathsf{H} = \hbar\omega(\mathsf{N} + \frac{1}{2}\mathsf{I})$.

Suppose $\mathsf{N}|\nu) = \nu|\nu)$. Then

$$\begin{aligned}\mathsf{a}^+\mathsf{N}|\nu) = \mathsf{a}^+(\mathsf{a}\mathsf{a}^+ - \mathsf{I})|\nu) = (\mathsf{N}\,\mathsf{a}^+ - \mathsf{a}^+)|\nu) &= \nu\,\mathsf{a}^+|\nu)\\ &\Downarrow\\ \mathsf{N}\,\mathsf{a}^+|\nu) &= (\nu+1)\mathsf{a}^+|\nu)\end{aligned}$$

and similarly

$$\mathsf{N}\,\mathsf{a}\;|\nu) = (\nu-1)\mathsf{a}\;|\nu)$$

Therefore $\mathsf{N}\,\mathsf{a}^p|\nu) = (\nu-p)\mathsf{a}^p|\nu) : p = 1, 2, 3, \ldots$ But N is positive semi-definite: if $|\beta) = \mathsf{a}|\alpha)$ then $(\alpha|\mathsf{N}|\alpha) = \||\beta)\|^2 \geqslant 0$ (all $|\alpha)$). To truncate the descending series $(\nu - p) : p = 1, 2, 3, \ldots$ we are forced to take ν to be an integer and to posit the existence of a ground state ("vacuum" state) $|0)$ such that

$$\mathsf{a}|0) = 0 \tag{4}$$

Building on that foundation, we have[3]

$$\begin{aligned}|1) &= c_0\,\mathsf{a}^+|0)\\ |2) &= c_1\,\mathsf{a}^+|1)\\ |3) &= c_2\,\mathsf{a}^+|2)\\ &\vdots\\ |n+1) &= c_n\,\mathsf{a}^+|n)\\ &\vdots\end{aligned}$$

To evaluate the constants c_n (which can without loss of generality be assumed to be real) we proceed

$$\left.\begin{aligned}(n|\mathsf{a}\,\mathsf{a}^+|n) &= (n+1)(n|n) = n+1\\ &= c_n^{-2}(n+1|n+1) = c_n^{-2}\end{aligned}\right\} \Longrightarrow c_n = \tfrac{1}{\sqrt{n+1}}$$

It now follows that

$$\begin{aligned}|n) &= \tfrac{1}{\sqrt{n}}\,\mathsf{a}^+|n-1)\\ &= \tfrac{1}{\sqrt{n(n-1)}}(\mathsf{a}^+)^2|n-2)\\ &\;\;\vdots\\ &= \tfrac{1}{\sqrt{n!}}(\mathsf{a}^+)^n|0)\end{aligned} \tag{5}$$

For the purposes at hand these results are most conveniently written

$$\mathsf{a}|n) = g_n|n-1), \;\; \mathsf{a}^+|n) = g_{n+1}|n+1) \quad\text{with}\quad g_n = \sqrt{n}$$

from which we recover $\mathsf{N}|n) = \mathsf{a}^+\mathsf{a}|n) = g_n\,\mathsf{a}^+|n-1) = g_n g_n|n) = n|n)$. If

[3] Here I borrow from my "Simultaneous measurement of noncommuting observables" (October 2012), page 5.

quantum oscillators were the objects of interest we could, by $\mathsf{H} = \hbar\omega(\mathsf{N} + \frac{1}{2}\mathsf{I})$, write

$$\mathsf{H}|n) = E_n|n) \quad \text{with} \quad E_n = \hbar\omega\big(\mathsf{n} + \tfrac{1}{2}\big)$$

and easily recover[4] descriptions of the familiar oscillator eigenstates

$$\psi_n(q) = (q|n)$$

But our present interest lies elsewhere.

Eigenstates of creation/annihilation operators: "coherent states". We want to describe the solutions of

$$\mathsf{a}|\alpha) = \alpha|\alpha) \quad \text{equivalently} \quad (\alpha|\mathsf{a}^+ = (\alpha|\bar{\alpha}$$

where by the non-hermiticity of a we expect the eigenvalues α to be complex. The states $|\alpha)$ are, for reasons that will emerge, called "coherent states." Writing

$$|\alpha) = \sum_{n=0}^{\infty} A_n|n)$$

we have

$$\mathsf{a}|\alpha) = \sum A_n\sqrt{n}\,|n-1) = \alpha\sum A_n|n)$$

giving $A_n\sqrt{n} = \alpha A_{n-1}$ whence

$$A_n = \frac{\alpha}{\sqrt{n}}A_{n-1} = \frac{\alpha^2}{\sqrt{n(n-1)}}A_{n-2} = \cdots = \frac{\alpha^n}{\sqrt{n!}}A_0$$

Therefore

$$|\alpha) = A_0\sum\frac{\alpha^n}{\sqrt{n!}}|n)$$

Normalization requires

$$1 = (\alpha|\alpha) = |A_0|^2\sum_m\sum_n\frac{\bar{\alpha}^m\alpha^n}{\sqrt{m!n!}}(m|n) = |A_0|^2\sum_n\frac{|\alpha|^{2n}}{n!} = |A_0|^2 e^{|\alpha|^2}$$

and supplies

$$A_0 = e^{-\frac{1}{2}|\alpha|^2}\cdot e^{i(\text{arbitrary phase})}$$

Discarding the phase factors, we have

$$|\alpha) = e^{-\frac{1}{2}|\alpha|^2}\sum_n\frac{\alpha^n}{\sqrt{n!}}|n) \tag{6}$$

where α ranges freely on the complex plane.

Let α and β mark two points on the complex plane. Immediately

[4] See Griffiths, §2.3.1 or *Advanced Quantum Topics*, Chapter 0, pages 40-41.

$$
\begin{aligned}
(\beta|\alpha) &= e^{-\frac{1}{2}|\beta|^2-\frac{1}{2}|\alpha|^2}\sum_m\sum_n\frac{\bar\beta^m\alpha^n}{\sqrt{m!n!}}(m|n)\\
&= e^{-\frac{1}{2}|\beta|^2-\frac{1}{2}|\alpha|^2}\sum_n\frac{\bar\beta^n\alpha^n}{n!}\\
&= e^{-\frac{1}{2}|\beta|^2-\frac{1}{2}|\alpha|^2+\bar\beta\alpha}
\end{aligned}
\tag{7.1}
$$

Observing that

$$
\begin{aligned}
-\tfrac{1}{2}|\beta|^2-\tfrac{1}{2}|\alpha|^2+\bar\beta\alpha &= -\tfrac{1}{2}\bar\beta\beta-\tfrac{1}{2}\bar\alpha\alpha+\tfrac{1}{2}\bar\beta\alpha+\tfrac{1}{2}\beta\bar\alpha\\
&\qquad+\tfrac{1}{2}\bar\beta\alpha-\tfrac{1}{2}\beta\bar\alpha\\
&= -\tfrac{1}{2}|\beta-\alpha|^2+\tfrac{1}{2}(\bar\beta\alpha-\beta\bar\alpha)
\end{aligned}
$$

we have

$$
(\beta|\alpha) = R(\beta,\alpha)\,e^{i\varphi(\beta,\alpha)} \tag{7.2}
$$

where $R(\beta,\alpha)=\exp\big\{-\frac{1}{2}|\beta-\alpha|^2\big\}$ and $i\varphi(\beta,\alpha)=\frac{1}{2}(\bar\beta\alpha-\beta\bar\alpha)$. So

$$
\begin{aligned}
(\beta|\alpha) &\neq 0\\
&\approx 0 \quad\text{when } |\beta-\alpha| \text{ is large}
\end{aligned}
$$

But while coherent states—though normalized—are not orthogonal, they are complete in the sense

$$
\int|\alpha)\frac{d^2\alpha}{\pi}(\alpha| = \mathbf{I} \quad : \quad d^2\alpha = d\alpha_r d\alpha_i \tag{8}
$$

as I demonstrate:[5] We have

$$
\int|\alpha)\frac{d^2\alpha}{\pi}(\alpha| = \frac{1}{\pi}\sum_m\sum_n\frac{1}{\sqrt{m!n!}}\bigg\{\int e^{-|\alpha|^2}\bar\alpha^m\alpha^n d^2\alpha\bigg\}|m)(n|
$$

In polar coordinates $\alpha = re^{i\theta}$ and $d^2\alpha = rdrd\theta$ so we have

$$
\big\{\text{etc.}\big\} = \int_0^\infty e^{-r^2}r^{m+n+1}dr\cdot\int_0^{2\pi}e^{i(n-m)\theta}d\theta = 2\pi\delta_{mn}\int_0^\infty e^{-r^2}r^{m+n+1}dr
$$

giving

$$
\int|\alpha)\frac{d^2\alpha}{\pi}(\alpha| = \sum_n\frac{2}{n!}\int_0^\infty e^{-r^2}r^{2n+1}dr|n)(n| = \sum_n|n)(n| = \mathbf{I}
$$

where we have used $2\int_0^\infty e^{-r^2}r^{2n+1}dr = \Gamma(n+1) = n!$. While this establishes

[5] Here again, my source is Gerry & Knight's §3.5.

completeness, the coherent states $|\alpha)$ are in fact "overcomplete." For we have

$$\begin{aligned}|\beta) &= \int |\alpha)\frac{d^2\alpha}{\pi}(\alpha|\beta) \\ &= \int \frac{R(\alpha,\beta)\,e^{i\varphi(\alpha,\beta)}}{\pi}|\alpha)\,d^2\alpha = \text{weighted superposition of } |\alpha) \text{ states} \quad (9)\end{aligned}$$

The "reproducing kernel"

$$K(\alpha,\beta) = \frac{R(\alpha,\beta)\,e^{i\varphi(\alpha,\beta)}}{\pi} = \tfrac{1}{\pi}\,e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\bar{\alpha}\beta}$$

stands where a delta function $\delta(\alpha-\beta)$ would stand if the coherent satates were linearly independent. Notice that $|\beta)$ itself contributes—but only fractionally—to the preceding representation of $|\beta)$: $K(\beta,\beta) = \frac{1}{\pi}$. "Overcomplete bases" are not commonly encountered, but are not intrinsically bizarre: look, for example, to the quartet of normalized 2-vectors

$$\mathbf{e}_1 = \begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{e}_2 = \begin{pmatrix}0\\1\end{pmatrix},\ \mathbf{e}_3 = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix},\ \mathbf{e}_4 = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$$

which are seen to be complete in a sense

$$\sum_{j=1}^{4}\frac{\mathbf{e}_j\,\mathbf{e}_j^+}{2} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

that entails

$$\mathbf{e}_k = \sum_j \mathbf{e}_j K_{jk} \quad \text{with} \quad K_{jk} = \tfrac{1}{2}(\mathbf{e}_j,\mathbf{e}_k)$$

The fractional contribution of $\mathbf{e}_k$ itself to the sum enters with weight $K_{kk} = \frac{1}{2}$. The model fails, however, in one respect: the coherent states $|\alpha)$ arose as eigenstates of a linear operator $\mathbf{a}$, but the vectors $\mathbf{e}_k$ cannot be produced as eigenvectors of a matrix $\mathbb{A}$; the matrices $\mathbb{P}_j = \frac{1}{2}\mathbf{e}_j\,\mathbf{e}_j^+$ are complete but are neither projective ($\mathbb{P}_j\mathbb{P}_j = \frac{1}{2}\mathbb{P}_j$) nor orthogonal. Spectral decomposition methods are evidently not available when one is dealing with overcomplete bases.

We cannot expect the transformation from the complete orthonormal basis $\{|n)\}$ to/from the overcomplete non-orthogonal basis to be unitary (or even to be "square"). And indeed,

$$\int (m|\alpha)\frac{d^2\alpha}{\pi}(\alpha|n) = (m|n) = \delta_{mn}$$

$$\sum_n (\alpha|n)(n|\beta) = (\alpha|\beta) = R(\alpha,\beta)\,e^{i\varphi(\alpha,\beta)} \neq \delta(\alpha-\beta)$$

so while $(\alpha|n)$ possesses a left inverse, it does not possess a right inverse. For a finite-dimensional instance of such a situation consider the matrix

$$\mathbb{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which supplies

$$\mathbb{U}^{+}\mathbb{U} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{U}\mathbb{U}^{+} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Coherent representation of states and operators. Let ψ_n denote the coordinates of $|\psi)$ in the number basis:

$$|\psi) = \sum_n |n)\psi_n \quad \text{with} \quad \psi_n = (n|\psi), \ \sum_n \bar{\psi}_n\psi_n = 1$$

Passing to the coherent basis, we have

$$|\psi) = \tfrac{1}{\pi}\int |\alpha)d^2\alpha(\alpha|\psi) = \tfrac{1}{\pi}\int |\alpha)d^2\alpha \sum_n \psi_n(\alpha|n)$$

From the adjoint of (6) we obtain

$$(\alpha|n) = e^{-\frac{1}{2}|\alpha|^2}\sum_m \frac{\bar{\alpha}^m}{\sqrt{m!}}(m|n) = e^{-\frac{1}{2}|\alpha|^2}\frac{\bar{\alpha}^n}{\sqrt{n!}}$$

giving

$$|\psi) = \tfrac{1}{\pi}\int |\alpha)d^2\alpha\Big\{e^{-\frac{1}{2}|\alpha|^2}\psi(\bar{\alpha})\Big\} \tag{10.1}$$

$$\psi(\,z\,) = \sum_n \psi_n \frac{z^n}{\sqrt{n!}} \tag{10.2}$$

Here $\big\{\text{etc.}\big\}$ is the α^{th} coherent coordinate of $|\psi)$ and $\psi(z)$ is an element of the "Bargmann-Segal space" of entire functions.[6]

[6] V. Bargmann, "On a Hilbert space of analytic functions and an associated integral transform," Comm. Pure & Applied Math. **14**, 187-214 (1961) and "Remarks on a Hilbert space of analytic functions," PNAS **48**, 199-203 (1962); I. Segal, "Mathematical characterization of the physical vacuum for a linear Bose-Einstein field," Illinois J. Math. **6**, 500-523 (1962). In "On the applications of Bargmann Hilbert spaces to dynamical problems," Annals of Physics **41**, 205-229 (1967) S. Schweber illustrates the utility of Bargmann-Segal concepts in quantum contexts that have nothing to do with coherent radiation (but much to do with oscillators.)

Turning from states to operators... in the number basis a linear operator F—not presently assumed to be hermitian—acquires the representation

$$\mathsf{F} = \sum_{m,n} |m)F_{mn}(n| \quad \text{with} \quad F_{mn} = (m|\mathsf{F}|n), \text{ elements of } \mathbb{F}$$

In particular, we have

$$\mathbb{N} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and use $\mathsf{a}|n) = \sqrt{n}\,|n-1)$, $\mathsf{a}^+|n) = \sqrt{n+1}\,|n-1)$ to obtain

$$\mathbb{A} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \mathbb{A}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In the number representation the operators

$$\mathsf{q} = \sqrt{\hbar/2}(\mathsf{a}^+ + \mathsf{a}), \quad \mathsf{p} = i\sqrt{\hbar/2}(\mathsf{a}^+ - \mathsf{a})$$

therefore become

$$\mathbb{Q} = \sqrt{\hbar/2}\begin{pmatrix} 0 & +\sqrt{1} & 0 & 0 & \cdots \\ \sqrt{1} & 0 & +\sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & +\sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathbb{P} = i\sqrt{\hbar/2}\begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

which are seen to be hermitian. The fundamental commutator $[\mathsf{q}, \mathsf{p}] = i\hbar\,\mathsf{I}$ does not admit of finite-dimensional representation, since $\text{tr}[\mathbb{Q}, \mathbb{P}] = 0$ in all finite-dimensional cases, while $\text{tr}[i\hbar\,\mathsf{I}] = i\hbar\infty$; for the matrices described above we in (for example) the 5-dimensional case obtain

$$[\mathbb{Q}, \mathbb{P}] = i\hbar \cdot \text{diag}\{1, 1, 1, 1, -4\} \neq i\hbar\mathbb{I}$$

which is traceless. In the ∞-dimensional case the final term is "pushed out of the picture" and (formally!) we recover $[\mathbb{Q}, \mathbb{P}] = i\hbar\,\mathbb{I}$.

Passing from the number basis to the coherent basis, we have

$$\begin{aligned}\mathsf{F} &= \sum_{m,n} |m)F_{mn}(n| \\ &= \tfrac{1}{\pi^2}\iint \sum_{m,n} |\alpha)d^2\alpha(\alpha|m)F_{mn}(n|\beta)d^2\beta(\beta| \\ &= \tfrac{1}{\pi^2}\iint |\alpha)d^2\alpha\Big\{e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\mathcal{F}(\bar{\alpha},\beta)\Big\}d^2\beta(\beta| \end{aligned}\tag{11.1}$$

where

$$\begin{aligned}\mathcal{F}(\bar{\alpha},\beta) &= \sum_{mn} F_{mn}\frac{\bar{\alpha}^m\beta^n}{\sqrt{m!n!}} \\ &= e^{\frac{1}{2}(|\alpha|^2+|\beta|^2)}(\alpha|\,\mathsf{F}\,|\beta)\end{aligned}\tag{11.2}$$

On the diagonal

$$e^{|\alpha|^2}(\alpha|\,\mathsf{F}\,|\alpha) = \mathcal{F}(\bar{\alpha},\alpha) = \sum_{mn} F_{mn}\frac{\bar{\alpha}^m\alpha^n}{\sqrt{m!n!}}$$

Since the $|\alpha)$-basis is overcomplete, it is perhaps not surprising that the diagonal elements $(\alpha|\,\mathsf{F}\,|\alpha)$ are sufficient to determine all the matrix elements $(m|\,\mathsf{F}\,|n)$:

$$F_{mn} = \frac{1}{\sqrt{m!n!}}\Big(\frac{\partial}{\partial\bar{\alpha}}\Big)^m\Big(\frac{\partial}{\partial\alpha}\Big)^n\mathcal{F}(\bar{\alpha},\alpha)$$

Look to the coherent description of the trace. We have

$$\begin{aligned}\mathrm{tr}\,\mathsf{F} = \sum_n (n|\,\mathsf{F}\,|n) &= \tfrac{1}{\pi^2}\sum_n \iint (n|\alpha)d^2\alpha(\alpha|\,\mathsf{F}\,|\beta)d^2\beta(\beta|n) \\ &= \tfrac{1}{\pi^2}\iint \sum_n d^2\alpha(\alpha|\,\mathsf{F}\,|\beta)d^2\beta(\beta|n)(n|\alpha) \\ &= \tfrac{1}{\pi^2}\iint d^2\alpha(\alpha|\,\mathsf{F}\,|\beta)d^2\beta(\beta|\alpha)\end{aligned}$$

But the overcompleteness relation reads

$$\tfrac{1}{\pi}\int |\beta)d^2\beta(\beta|\alpha) = |\alpha)$$

so we have

$$\mathrm{tr}\,\mathsf{F} = \tfrac{1}{\pi}\int (\alpha|\,\mathsf{F}\,|\alpha)d^2\alpha \tag{12}$$

which is—satisfyingly—quite as expected.

Evaluation of $(\alpha|\mathsf{F}|\beta)$ becomes trivial when F is presented as an $\mathsf{a}^+\mathsf{a}$-ordered function of a^+ and a

$$\mathsf{F} = {}_{\mathsf{a}^+}\Big[F(a^+,a)\Big]_{\mathsf{a}}$$

for then

$$(\alpha|\mathsf{F}|\beta) = F(\alpha,\beta)\cdot(\alpha|\beta) = F(\alpha,\beta)\cdot e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2+\bar\alpha\beta} \tag{13}$$

Look, for example, to the unitary "displacement operator"[7]

$$\mathsf{D}(u) = \exp\big\{u\,\mathsf{a}^+ - \bar u\,\mathsf{a}\big\} \tag{14}$$

When A and B commute with their commutator $\mathsf{C} \equiv [\mathsf{A},\mathsf{B}]$ we have the well-known Kermack-McCrae identity[8]

$$e^{\mathsf{A}+\mathsf{B}} = \begin{cases} e^{-\frac{1}{2}\mathsf{C}}\cdot e^{\mathsf{A}}e^{\mathsf{B}} & : \quad \mathsf{AB}\text{-ordered} \\ e^{+\frac{1}{2}\mathsf{C}}\cdot e^{\mathsf{B}}e^{\mathsf{A}} & : \quad \mathsf{BA}\text{-ordered}\end{cases} \tag{15}$$

From $[u\,\mathsf{a}^+, -\bar u\,\mathsf{a}] = -|u|^2[\mathsf{a}^+,\mathsf{a}] = +|u|^2\,\mathsf{I}$ we are led therefore to

$$\begin{aligned}\mathsf{D}(u) &= e^{-\frac{1}{2}|u|^2}\exp\big\{u\,\mathsf{a}^+\big\}\exp\big\{-\bar u\,\mathsf{a}\big\}\\ &\Downarrow\\ (\alpha|\mathsf{D}(u)|\beta) &= e^{-\frac{1}{2}|u|^2+u\bar\alpha-\bar u\beta}\end{aligned}$$

which provides a concrete instance of (13). More to the point and more interestingly: it is clear that

$$\mathsf{a}|0) = 0 \;\Rightarrow\; \exp\big\{-\bar u\,\mathsf{a}\big\}|0) = \sum_{k=0}^\infty(-)^k\frac{1}{k!}\bar u^2\,\mathsf{a}^k|0) = \mathsf{a}^0|0) = |0)$$

so we have

$$\begin{aligned}\mathsf{D}(u)|0) &= e^{-\frac{1}{2}|u|^2}\exp\big\{u\,\mathsf{a}^+\big\}|0)\\ &= e^{-\frac{1}{2}|u|^2}\sum_{n=0}^\infty\frac{u^n}{n!}(\mathsf{a}^+)^n|0)\\ &= e^{-\frac{1}{2}|u|^2}\sum_{n=0}^\infty\frac{u^n}{\sqrt{n!}}|n) && \text{by (5)}\\ &= |u) && \text{by (6)}\end{aligned} \tag{16.1}$$

at which point it becomes clear why $\mathsf{D}(u)$ operators are called "displacement operators." This result serves to establish the sense in which

$$|\alpha) = \mathsf{D}(\alpha)|0) \quad \text{is a "displaced vacuum state"} \tag{16.2}$$

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[7] Roy J. Glauber, "Coherent and incoherent states of the radiation field," Phys. Rev. **131**, 2766-2788 (1963), §III.

[8] *Advanced Quantum Topics* (2000), Chapter 0, page 32.

Products of unitaries are unitary; looking to the product $\mathsf{D}(u)\,\mathsf{D}(v)$, we have

$$\mathsf{D}(u)\,\mathsf{D}(v) = \exp\big\{u\,\mathsf{a}^+ - \bar{u}\,\mathsf{a}\big\}\exp\big\{v\,\mathsf{a}^+ - \bar{v}\,\mathsf{a}\big\}$$

Appealing again to (15) with $\mathsf{C} = [u\,\mathsf{a}^+ - \bar{u}\,\mathsf{a}, v\,\mathsf{a}^+ - \bar{v}\,\mathsf{a}] = (u\bar{v} - \bar{u}v)\,\mathsf{I}$ we find

$$\mathsf{D}(u)\,\mathsf{D}(v) = e^{\frac{1}{2}(u\bar{v}-\bar{u}v)}\cdot\mathsf{D}(u+v) \tag{17}$$

Since the multiplier $e^{\frac{1}{2}(u\bar{v}-\bar{u}v)}$ is of the form $e^{i\phi}$ it is physically inconsequential, but ϕ admits nevertheless of an interesting geometrical interpretation: if $u = re^{i\rho}$ and $v = se^{i\sigma}$ then $\phi = rs\sin(\rho - \sigma)$. Notice also that

$$\mathsf{D}^+(u) = \mathsf{D}^{-1}(u) = \mathsf{D}(-u) \tag{18}$$

Minimum uncertainty states. The Schrödinger inequality[9]

$$(\Delta A)^2(\Delta B)^2 \geqslant \Big\langle\frac{\mathsf{A\,B} - \mathsf{B\,A}}{2i}\Big\rangle^2 + \Big[\Big\langle\frac{\mathsf{A\,B} + \mathsf{B\,A}}{2}\Big\rangle - \langle\mathsf{A}\rangle\langle\mathsf{B}\rangle\Big]^2 \tag{19}$$

becomes

$$(\Delta x)^2(\Delta p)^2 \geqslant \Big(\frac{\hbar}{2}\Big)^2 + \Big[\Big\langle\frac{\mathsf{x\,p} + \mathsf{p\,x}}{2}\Big\rangle - \langle\mathsf{x}\rangle\langle\mathsf{p}\rangle\Big]^2$$

in the most familar instance. Minimal uncertainty $\Delta x\Delta p = \frac{1}{2}\hbar$ is achieved if and only if $[etc.] \equiv$ "quantum correlation coefficient"[10] $= 0$. Familiarly, the underlying Cauchy-Schwarz inequality $(f|f)(g|g) \geqslant |(f|g)|^2$ becomes equality if and only if either $|f)$ else $|g) = 0$ or $|g) = \lambda|f)$. In the present application $|f) = 0$ becomes $\mathsf{A}|\psi) = \langle\mathsf{A}\rangle|\psi)$, so $|\psi)$ is an eigenstate of A and (19) is reduced to a triviality: $\Delta A = 0$ (else $\Delta B = 0$). More interesting are the implications[11] of $|g) = \lambda|f)$, which now reads

$$\big(\mathsf{B} - \mathsf{b}\big)|\psi) = \lambda\big(\mathsf{A} - \mathsf{a}\big)|\psi) \quad\text{with}\quad \mathsf{a} = \langle\mathsf{A}\rangle\,\mathsf{I},\;\; \mathsf{b} = \langle\mathsf{B}\rangle\,\mathsf{I} \tag{20}$$

Multiplication by $(\mathsf{A} - \mathsf{a})$ else $(\mathsf{B} - \mathsf{b})$ gives

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[9] See "Simultaneous measurement of noncommuting quantum observables" (October 2012), page 3.

[10] In classical statistics, if x and y are random variables then one has (for all m and n)

$$\langle x^m y^n\rangle = \langle x^m\rangle\langle y^n\rangle \quad\text{iff } x \text{ ane } y \text{ are statistically independent}$$

The number $\langle x\,y\rangle - \langle x\rangle\langle y\rangle$ provides therefore a leading indicator of the extent to which x and y are statistically dependent or correlated. Generally $\mathsf{A\,B} \neq \mathsf{B\,A}$. On the right side of (19) we are told to "split the difference." See page 202 in D. Bohm, *Quantum Mechanics* (1951).

[11] See E. Merzbacher, *Quantum Mechanics* (2nd edition, 1970), page 160.

$$\lambda(\Delta A)^2 = \big\langle(\mathbf{A}-\mathsf{a})(\mathbf{B}-\mathsf{b})\big\rangle = \langle\mathbf{A}\,\mathbf{B}\rangle - \langle\mathbf{A}\rangle\langle\mathbf{B}\rangle$$
$$\lambda^{-1}(\Delta B)^2 = \big\langle(\mathbf{B}-\mathsf{b})(\mathbf{A}-\mathsf{a})\big\rangle = \langle\mathbf{B}\,\mathbf{A}\rangle - \langle\mathbf{A}\rangle\langle\mathbf{B}\rangle$$

which when added/subtracted give

$$\lambda(\Delta A)^2 + \lambda^{-1}(\Delta B)^2 = 2\Big[\Big\langle\frac{\mathbf{A}\,\mathbf{B}+\mathbf{B}\,\mathbf{A}}{2}\Big\rangle - \langle\mathbf{A}\rangle\langle\mathbf{B}\rangle\Big] \;:\; \text{want} = 0 \tag{21.1}$$
$$\lambda(\Delta A)^2 - \lambda^{-1}(\Delta B)^2 = i\langle\mathbf{C}\rangle \tag{21.2}$$

where the hermitian operator $\mathbf{C} \equiv \frac{1}{i}(\mathbf{A}\,\mathbf{B} - \mathbf{B}\,\mathbf{A})$. The first of those equations supplies $\lambda^2 = -(\Delta B/\Delta A)^2$ so λ is necessarily imaginary: $\lambda = \pm i(\Delta B/\Delta A)$. Returning with this information to (21.2) we obtain

$$\big(\Delta A\,\Delta B\big)_{\text{min}} = \pm\tfrac{1}{2}\langle\mathbf{C}\rangle \quad : \quad \text{discard the absurd sign}$$

Reverting again to the case $\mathbf{A} = \mathbf{x}$, $\mathbf{B} = \mathbf{p}$ we recover $\big(\Delta x\,\Delta p\big)_{\text{min}} = \frac{1}{2}\hbar$. In this instance (20) reads $\lambda^{-1}(\mathbf{p} - p_0\mathbf{I})|\psi) = (\mathbf{x} - x_0\mathbf{I})|\psi)$ which can be written

$$(\mathbf{x} - \lambda^{-1}\mathbf{p})|\psi) = (x_0 - \lambda^{-1}p_0)|\psi)$$

where $x_0 = \langle\mathbf{x}\rangle$, $p_0 = \langle\mathbf{p}\rangle$, $\lambda^{-1} = -i\,\Delta x/\Delta p$. In the x-representation

$$\big(\tfrac{\hbar}{i}\partial_x - p_0\big)\psi(x) = \lambda(x - x_0)\psi(x) \quad : \quad \lambda = i\,\frac{\Delta p}{\Delta x} = i\,\frac{\hbar}{2(\Delta x)^2}$$
$$\Downarrow$$
$$\partial_x\psi(x) = \Big[-\frac{x - x_0}{2(\Delta x)^2} + i\,p_0x/\hbar\Big]\psi(x)$$

of which the normalized solution is

$$\psi(x) = \Big(\frac{1}{2\pi(\Delta x)^2}\Big)^{\frac{1}{4}}\exp\Big\{-\Big(\frac{x - x_0}{2\,\Delta x}\Big)^2 + i\,p_0x/\hbar\Big\} \tag{22.1}$$

Such "launched Gaussian" states were first encountered by Schrödinger in the paper[12] in which he reported the Schrödinger inequality (19), and in my own writing were most recently encountered[13] as states produced by Arthurs-Kelly measurements when the detectors read $\{x_0, p_0\}$. At $x_0 = p_0 = 0$ we have

$$\psi_0(x) = \Big(\frac{1}{2\pi(\Delta x)^2}\Big)^{\frac{1}{4}}\exp\Big\{-\Big(\frac{x}{2\,\Delta x}\Big)^2\Big\} \tag{22.2}$$
$$= \sqrt{\text{centered normal distribution with standard deviation } \Delta x}$$

---

[12] "Zur Heisenbergschen Unshärfeprinzip," Berliner Berichte, 296–303 (1930)

[13] See equation (12.1) on page 17 of [9]. In such measurements the value of Δx (as also of $\Delta p = \frac{1}{2}\hbar/\Delta x$) is set by the "balanced Gaussian" initial states of the detectors.

which becomes a description of the oscillator ground state when Δx is assigned a value set by the system parameters: $\Delta x = \sqrt{2m\omega/\hbar}$.

More abstractly, we have $\psi_0(x) = (x|0)$ where $(0|0) = 1$ and $(\mathbf{x}+i\beta\,\mathbf{p})|0) = 0$. The latter equation can be written in a variety of ways, of which for our purposes the most interesting is $\mathbf{a}|0) = 0$ where $\mathbf{a}$ is the dimensionless non-hermitian operator defined

$$\mathbf{a} = \sqrt{\frac{\Delta x \Delta p}{2\hbar}}\left(\frac{\mathbf{x}}{\Delta x} + i\frac{\mathbf{p}}{\Delta p}\right) = \tfrac{1}{\sqrt{2\hbar}}(\xi^{-1}\mathbf{x} + i\xi\,\mathbf{p}) \qquad (23)$$
$$\xi = \sqrt{\Delta x/\Delta p}$$

Here again, $[\mathbf{x},\mathbf{p}] = i\hbar\mathbf{I} \;\Rightarrow\; [\mathbf{a},\mathbf{a}^+] = \mathbf{I}$ so we have been led back—this time not from oscillator theory but from Schrödinger's identity—to the operators $\{\mathbf{a},\mathbf{a}^+\}$ that support what is called the theory of coherent states but on the basis of the preceding argument might more reasonably be called the theory of *minimal uncertainty* states.

The eigenstates $\mathbf{a}$ (which I will for the purposes of this discussion denote $|u)$ because the symbol α will have other work to do) can be obtained by "displacement" of the vacuum/ground state $|0)$, as was remarked at (16). In the x-representation we have

$$\psi_u(x) = (x|\mathbf{D}(u)|0)$$

where

$$\begin{aligned}
\mathbf{D}(u) &= \exp\left\{u\,\mathbf{a}^+ - \bar{u}\,\mathbf{a}\right\}\\
&= \exp\left\{\tfrac{1}{\sqrt{2\hbar}}u(\xi^{-1}\mathbf{x} - i\xi\,\mathbf{p}) - \tfrac{1}{\sqrt{2\hbar}}\bar{u}(\xi^{-1}\mathbf{x} + i\xi\,\mathbf{p})\right\}\\
&\downarrow\\
&= \exp\left\{\tfrac{i}{\hbar}(\alpha\,\mathbf{p} + \beta\,\mathbf{x})\right\} \quad \text{if we write } u = \tfrac{1}{\sqrt{2\hbar}}(-\xi^{-1}\alpha + i\xi\,\beta)\\
&= \exp\left\{\tfrac{1}{2}\tfrac{i}{\hbar}\alpha\beta\right\}\cdot\exp\left\{\tfrac{i}{\hbar}\beta\,\mathbf{x}\right\}\exp\left\{\tfrac{i}{\hbar}\alpha\,\mathbf{p}\right\}
\end{aligned}$$

supplies

$$\begin{aligned}
\psi_u(x) &= \exp\left\{\tfrac{1}{2}\tfrac{i}{\hbar}\alpha\beta\right\}\cdot\exp\left\{\tfrac{i}{\hbar}\beta x\right\}\exp\left\{\alpha\partial_x\right\}\psi_0(x)\\
&= \exp\left\{\tfrac{1}{2}\tfrac{i}{\hbar}\alpha\beta\right\}\cdot\exp\left\{\tfrac{i}{\hbar}\beta x\right\}\psi_0(x+\alpha)\\
&= \exp\left\{\tfrac{1}{2}\tfrac{i}{\hbar}\alpha\beta\right\}\cdot\left(\frac{1}{2\pi(\Delta x)^2}\right)^{\frac{1}{4}}\exp\left\{-\left(\frac{x+\alpha}{2\,\Delta x}\right)^2 + i\beta x/\hbar\right\}
\end{aligned}$$

If we discard the physically irrelevant phase factor $\exp\left\{\tfrac{1}{2}\tfrac{i}{\hbar}\alpha\beta\right\}$ and set $\alpha = -x_0$, $\beta = p_0$—which is in effect to set

$$u = \tfrac{1}{\sqrt{2\hbar}}(\xi^{-1}x_0 + i\xi\,p_0)$$

(and could, in view of (23), not be more natural)—we recover precisely (22.1).

Some additional means and uncertainties. The vacuum state $|0)$—which by one interpretation became the oscillator ground state, and was defined by an equation $\mathsf{a}|0) = 0$ that implies $\mathsf{N}|0) = 0$—arose from the requirement that

$$\Delta x\,\Delta p = \text{minimum}$$

Specifically, we found $(0|\mathsf{x}|0) = (0|\mathsf{p}|0) = 0$ which gave

$$\Delta x\,\Delta p = (0|\mathsf{x}^2|0)\cdot(0|\mathsf{p}^2|0) = \tfrac{1}{2}\hbar$$

I look now to the evaluation of $(n|\mathsf{x}^2|n)\cdot(n|\mathsf{p}^2|n)$. From (23) we obtain

$$\begin{aligned} \mathsf{x} &= \sqrt{\hbar/2}\,\xi\ \ \ (\mathsf{a}^+ + \mathsf{a}) \\ \mathsf{p} &= i\sqrt{\hbar/2}\,\xi^{-1}(\mathsf{a}^+ - \mathsf{a}) \end{aligned}$$

Drawing upon $\mathsf{a}|n) = \sqrt{n}\,|n-1)$, $\mathsf{a}^+|n) = \sqrt{n+1}\,|n+1)$ and the orthogonality of the $|n)$-states we have $(n|\mathsf{x}|n) = (n|\mathsf{p}|n) = 0$, so

$$\Delta_n x = \sqrt{(n|\mathsf{x}^2|n)},\quad \Delta_n p = \sqrt{(n|\mathsf{p}^2|n)}$$

where

$$(n|\mathsf{x}^2|n) = (\hbar/2)\xi^2(n|\mathsf{a}^+\mathsf{a}^+ + \mathsf{a}^+\mathsf{a} + \mathsf{a}\,\mathsf{a}^+ + \mathsf{a}\,\mathsf{a}|n)$$

But $(n|\mathsf{a}^+\mathsf{a}^+|n) = (n|\mathsf{a}\,\mathsf{a}|n) = 0$ by orthogonality, while

$$(n|\mathsf{a}^+\mathsf{a} + \mathsf{a}\,\mathsf{a}^+|n) = (n|\mathsf{N} + \mathsf{I}|n) = n + 1$$

so we have

$$(n|\mathsf{x}^2|n) = (\hbar/2)\xi^2\ \ (n+1)$$

Similarly

$$(n|\mathsf{p}^2|n) = (\hbar/2)\xi^{-2}(n+1)$$

so

$$\Delta_n x\,\Delta_n p = \tfrac{n+1}{2}\,\hbar \tag{24}$$

which gives back the familiar result at $n = 0$. A simple adjustment $|n) \to |\alpha)$ of the preceding argument supplies

$$\begin{aligned} (\alpha|\mathsf{x}|\alpha) &= \sqrt{\hbar/2}\,\xi\ \ (\bar\alpha + \alpha) \\ (\alpha|\mathsf{p}|\alpha) &= \sqrt{\hbar/2}\,\xi^{-1}(\bar\alpha - \alpha) \\ (\alpha|\mathsf{x}^2|\alpha) &= (\hbar/2)\xi^2\ \ [(\bar\alpha + \alpha)^2 + 1] \\ (\alpha|\mathsf{p}^2|\alpha) &= (\hbar/2)\xi^{-2}[(\bar\alpha - \alpha)^2 + 1] \end{aligned}$$

If the system is in state $|\alpha)$ then

$$\langle n\rangle = (\alpha|\mathsf{N}|\alpha) = (\alpha|\mathsf{a}^+\mathsf{a}|\alpha) = \bar\alpha\alpha = |\alpha|^2 \tag{25}$$

$$(n|\alpha) = e^{-\frac{1}{2}|\alpha|^2}\sum_m \frac{\alpha^m}{\sqrt{m!}}(n|m) = e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\quad\text{by (6)}$$

$$\Downarrow$$

$$|(n|\alpha)|^2 = e^{-|\alpha|^2}\frac{\bar\alpha^n\alpha^n}{n!} = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!} \equiv \wp\big(n;\langle n\rangle\big) \tag{26}$$

Obviously $\sum_0^\infty \wp\big(n;\langle n\rangle\big) = 1$. The discrete distribution $\wp\big(n;\langle n\rangle\big)$ is precisely the *Poisson distribution* with parameter $\langle n\rangle$. Evaluation of the higher moments

$$m_\nu = \sum_{k=0}^{\infty} n^\nu \wp(n) = (\alpha|\,\mathbf{N}^\nu|\alpha)$$

poses an interesting operator-ordering problem, and leads to a result that can be described

$$m_\nu = T_\nu(\langle n\rangle)$$

where $T_\nu(x) = \sum_{k=1}^{\nu} S(\nu,k)x^k$ is a "Torchard polynomial"[14] and $S(\nu,k)$ is a "Stirling number of the second kind." Thus

$$\begin{aligned}
m_0 &= 1\\
m_1 &= \langle n\rangle\\
m_2 &= \langle n\rangle + \langle n\rangle^2\\
m_3 &= \langle n\rangle + 3\langle n\rangle^2 + \langle n\rangle^3\\
m_4 &= \langle n\rangle + 7\langle n\rangle^2 + 6\langle n\rangle^3 + \langle n\rangle^4\\
m_5 &= \langle n\rangle + 15\langle n\rangle^2 + 25\langle n\rangle^3 + 10\langle n\rangle^4 + \langle n\rangle^5
\end{aligned}$$

which at $\langle n\rangle = 1$ become "Bell numbers."[15] In particular, we have

$$\begin{aligned}
m_2 = (\alpha|\,\mathbf{N}^2|\alpha) &= (\alpha|\,\mathbf{a}^+\mathbf{a}\,\mathbf{a}^+\mathbf{a}\,|\alpha)\\
&= (\alpha|\,\mathbf{a}^+\mathbf{a}^+\mathbf{a}\,\mathbf{a} + \mathbf{a}^+\mathbf{a}\,|\alpha)\\
&= (\bar{\alpha}\alpha)^2 + (\bar{\alpha}\alpha) = \langle n\rangle^2 + \langle n\rangle
\end{aligned}$$

in agreement with a result just stated. These results supply

$$\Delta n = \sqrt{m_2 - m_1^2} = \sqrt{\langle n\rangle} \quad : \quad \text{variance} = \text{mean}$$

[14] Such polynomials are sometimes called "Bell polynomials," and are produced in *Mathematica* by the command `BellB[ν,x]` once one has installed the `Combinatorica` package.

[15] The Bell number B_ν (produced by the command `BellB[ν]`) describes the number of distinct partitions of a ν-element set. Look, for example, to the 3-element set $\{a,b,c\}$, of which the distinct partitions

$$\begin{aligned}
&\{(a),(b),(c)\}\\
&\{(a),(b,c)\}\\
&\{(b),(a,c)\}\\
&\{(c),(a,b)\}\\
&\{(a,b,c)\}
\end{aligned}$$

are $B_3 = 5$ in number.

so the fractional uncertainty becomes

$$\frac{\Delta n}{\langle n\rangle} = \langle n\rangle^{-\frac{1}{2}}$$

which decreases as the mean $\langle n\rangle = |\alpha|^2$ increases. Recall in this connection that (by the central limit theorem) as the mean becomes large the Poisson distribution goes over to the Gaussian (or normal) distribution with that same mean = variance:

$$\wp(n;m) \approx \frac{1}{\sqrt{2\pi m}}\exp\Big\{-\frac{(n-m)^2}{2m}\Big\} \quad : \quad \text{set } m = \langle n\rangle$$

Harmonic motion of coherent wavepackets. Look to the hermitian operator defined

$$\mathsf{H} = \hbar\omega(\mathsf{N} + \tfrac{1}{2}\mathsf{I}) \quad : \quad \mathsf{N} = \mathsf{a}^+\mathsf{a}$$

where a is given by (23). Some quick algebra supplies

$$\begin{aligned}\mathsf{H} &= \tfrac{1}{2}\omega(\xi^2\mathsf{p}^2 + \xi^{-2}\mathsf{x}^2)\\ &= \tfrac{1}{2m}(\mathsf{p}^2 + m^2\omega^2\mathsf{q}^2) \quad \text{if we set} \quad \xi^2 \equiv \Delta x/\Delta p = \tfrac{1}{m\omega}\\ &= \mathsf{H}_{\text{osc}}\end{aligned}$$

I once had occasion[16] to indicate how quantum dynamics could be formulated as a "theory of interactive moments," and by way of introduction looked to the oscillatory case. I digress to present a brief recapitulation of that discussion. Working in the Heisenberg picture, we look to the motion of $\langle\mathsf{p}\rangle = (\psi|\mathsf{p}|\psi)$, where $|\psi)$ is arbitrary:

$$\tfrac{d}{dt}\langle\mathsf{p}\rangle = \tfrac{1}{i\hbar}\langle[\mathsf{p},\mathsf{H}]\rangle = -m\omega^2\langle\mathsf{x}\rangle$$

at which point we have acquired an interest in the motion of $\langle\mathsf{x}\rangle$, so write

$$\tfrac{d}{dt}\langle\mathsf{x}\rangle = \tfrac{1}{i\hbar}\langle[\mathsf{x},\mathsf{H}]\rangle = \tfrac{1}{m}\langle\mathsf{p}\rangle$$

Those coupled first-order equations entail

$$\tfrac{d^2}{dt^2}\langle\mathsf{x}\rangle_t + \omega^2\langle\mathsf{x}\rangle_t = 0 \quad\Longrightarrow\quad \langle\mathsf{x}\rangle_t = \langle\mathsf{x}\rangle_{\max}\cos(\omega t+\delta)$$

Therefore $\frac{d}{dt}\langle\mathsf{p}\rangle = -m\omega^2\langle\mathsf{x}\rangle_{\max}\cos(\omega t+\delta)$ which gives

$$\langle\mathsf{p}\rangle_t = \langle\mathsf{p}\rangle_{\max}\sin(\omega t+\delta) \quad \text{with} \quad \langle\mathsf{p}\rangle_{\max} = m\omega\langle\mathsf{x}\rangle_{\max}$$

We have been brought thus by the simplest of means to the striking general conclusion that for *all* states $|\psi)$ the expectation values $\langle\mathsf{x}\rangle_t$ and $\langle\mathsf{p}\rangle_t$ oscillate with angular frequency ω (90° out of phase, with interrelated amplitudes), just as do the classical variables $\{x(t), p(t)\}$. If one looks similarly to the motion of $\langle\mathsf{p}^2\rangle$ one acquires an interest in the motion of $\langle\mathsf{C}\rangle$ with $\mathsf{C} = \frac{1}{2}(\mathsf{x}\,\mathsf{p} + \mathsf{p}\,\mathsf{x})$,

[16] *Advanced Quantum Topics* (2000), Chapter2, pages 51-60.

whence of $\langle \mathsf{D}\rangle$ with $\mathsf{D} = \frac{1}{m}\mathsf{p}^2 - m\omega^2\mathsf{x}^2$ and finds that

$$\begin{aligned}\tfrac{d}{dt}\langle \mathsf{C}\rangle &= \tfrac{1}{i\hbar}\langle[\mathsf{C}, \mathsf{H}]\rangle = \langle \mathsf{D}\rangle\\ \tfrac{d}{dt}\langle \mathsf{D}\rangle &= \tfrac{1}{i\hbar}\langle[\mathsf{D}, \mathsf{H}]\rangle = -4\omega^2\langle \mathsf{C}\rangle\end{aligned}$$

We infer that $\langle \mathsf{C}\rangle_t$ and $\langle \mathsf{D}\rangle_t$ both satisfy equations of the form

$$\tfrac{d^2}{dt^2}\langle \mathsf{A}\rangle_t + 4\omega^2\langle \mathsf{A}\rangle_t = 0$$

and therefore oscillate with doubled frequency:

$$\begin{aligned}\langle \mathsf{C}\rangle_t &= \quad C\sin(2\omega t+\delta)\\ \langle \mathsf{D}\rangle_t &= 2\omega C\cos(2\omega t+\delta)\end{aligned}$$

Integration of

$$\begin{aligned}\tfrac{d}{dt}\langle \mathsf{p}^2\rangle &= \tfrac{1}{i\hbar}\langle[\mathsf{p}^2, \mathsf{H}]\rangle = -2m\omega^2\langle \mathsf{C}\rangle\\ \tfrac{d}{dt}\langle \mathsf{x}^2\rangle &= \tfrac{1}{i\hbar}\langle[\mathsf{x}^2, \mathsf{H}]\rangle = \tfrac{1}{m}\langle \mathsf{C}\rangle\end{aligned}$$

now gives

$$\begin{aligned}\langle \mathsf{p}^2\rangle_t &= P^2 - m\omega\, C\sin(2\omega t+\delta)\\ \langle \mathsf{x}^2\rangle_t &= X^2 + \tfrac{1}{m\omega}C\sin(2\omega t+\delta)\end{aligned}$$

where P^2 and X^2 are constants of integration that by energy conservation are constrained to satisfy

$$\tfrac{1}{2m}\langle \mathsf{p}^2\rangle_t + \tfrac{1}{2}m\omega^2\langle \mathsf{x}^2\rangle_t = \tfrac{1}{2m}P^2 + \tfrac{1}{2}m\omega^2X^2 = E$$

In work previously cited[16] I discuss how—in principle—such arguments can be carried out with Hamiltonians of arbitrary design.

The point to which I draw particular attention is that while H_{osc} causes moments of all orders to oscillate with what might be called "classical rigidity," it is (in the Schrödinger picture) typically *not* the case that H_{osc} induces wavepackets to move rigidly:

$$\psi(x,t) \neq \text{ rigid translate of } \psi(x,0)$$

Generally

$$\begin{aligned}|\psi)_t = \mathsf{U}(t)|\psi)_0 \quad\text{with}\quad \mathsf{U}(t) &= \exp\big\{-\tfrac{i}{\hbar}\mathsf{H}_{\text{osc}}t\big\}\\ &= e^{-i\frac{1}{2}\omega t}\cdot\exp\big\{-i\omega t\,\mathsf{N}\big\}\end{aligned}$$

Clearly

$$|n)_t = e^{-i(n+\frac{1}{2})\,\omega t}\,|n)_0 \tag{27}$$

so the $|n)$-states (energy eigenstates) simply "sit there and buzz" with their respective frequencies. Writing $\psi_n(x,t) = (x|n)_t$ we have

$$\psi_n(x,t) = e^{-i(n+\frac{1}{2})\,\omega t}\,\psi_n(x,0) \quad\Longrightarrow\quad |\psi_n(x,t)|^2 = |\psi_n(x,0)|^2$$

The probability density $|\psi_n(x,t)|^2$ does in fact not move, but famously mimics the positional distribution of a classical oscillator with energy $E_n = (n+\frac{1}{2})\hbar\omega$.[17]

Coherent states were seen at (6) to be linear combinations of number states (*i.e.*, of energy eigenstates, each buzzing at its own frequency) and possess therefore the character of wavepackets. As it happens, the buzzing components conspire to produce a net motion which is in this instance quite distinctive. We have

$$|\alpha)_t = e^{-i\frac{1}{2}\omega t}\cdot\exp\big\{-i\omega t\,\mathsf{N}\big\}|\alpha)_0$$

From $\mathsf{N} = \mathsf{a}^+\mathsf{a}$ and $[\mathsf{a},\mathsf{a}^+] = \mathsf{I}$ we get $\mathsf{a}\,\mathsf{N}^k = (\mathsf{N}+\mathsf{I})^k\mathsf{a}\ :\ k = 0,1,2,3,\ldots$ whence

$$\mathsf{a}\cdot e^{u\mathsf{N}} = e^u e^{u\mathsf{N}}\,\mathsf{a}$$

which when applied to $|\alpha)$ gives

$$\mathsf{a}\cdot e^{u\mathsf{N}}|\alpha) = \alpha e^u\cdot e^{u\mathsf{N}}|\alpha) \quad\Longrightarrow\quad e^{u\mathsf{N}}|\alpha) = |\alpha e^u)$$

We therefore have

$$|\alpha)_t = e^{-i\frac{1}{2}\omega t}\cdot|\alpha e^{-i\omega t}) \tag{28}$$

The prefactor $e^{-i\frac{1}{2}\omega t}$ is unphysical; $\alpha e^{-i\omega t}$ traces with angular velocity ω a circle of radius $|\alpha|$ on the complex α-plane.

We established on page 12 that if we write $\alpha = \frac{1}{\sqrt{2\hbar}}(\xi^{-1}x_0 + i\,\xi\,p_0)$ then in x-representation

$$\begin{aligned}\psi_\alpha(x) &= (x|\alpha) = (x|\mathsf{D}(\alpha)|0)\\ &= \exp\big\{-\tfrac{1}{2}\tfrac{i}{\hbar}x_0p_0\big\}\cdot\Big(\frac{1}{2\pi(\Delta x)^2}\Big)^{\frac{1}{4}}\exp\Big\{-\frac{1}{4}\Big(\frac{x-x_0}{\Delta x}\Big)^2 + ip_0x/\hbar\Big\}\end{aligned} \tag{29}$$

Since this is a minimal-uncertainty state, we have $\Delta x\,\Delta p = \hbar/2$ whence

$$\xi^2 \equiv \frac{\Delta x}{\Delta p} = \frac{2(\Delta x)^2}{\hbar} \quad\Longrightarrow\quad \Delta x = \sqrt{\hbar/2}\,\xi$$

But we saw on page 15 that when we import oscillator parameters we have $\xi = 1/\sqrt{m\omega}$, so

$$\Delta x = \sqrt{\hbar/2m\omega}$$

The evolved oscillator coherent state (displaced ground state) is described

$$\psi_\alpha(x,t) = \psi_{\alpha(t)}(x)$$

where

$$\begin{aligned}\alpha(t) &= \tfrac{1}{\sqrt{2\hbar}}(\xi^{-1}x_0 + i\,\xi\,p_0)(\cos\omega t - i\sin\omega t)\\ &= \tfrac{1}{\sqrt{2\hbar}}\big\{(\xi^{-1}x_0\cos\omega t + \xi\,p_0\sin\omega t) + i\,(\xi\,p_0\cos\omega t - \xi^{-1}x_0\sin\omega t)\big\}\\ &= \tfrac{1}{\sqrt{2\hbar}}\big\{\xi^{-1}(x_0\cos\omega t + \xi^2p_0\sin\omega t) + i\,\xi\,(p_0\cos\omega t - \xi^{-2}x_0\sin\omega t)\big\}\\ &\equiv \tfrac{1}{\sqrt{2\hbar}}\big\{\xi^{-1}x(t) + i\,\xi\,p(t)\big\}\end{aligned}$$

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[17] See FIGURE 2.7(b) on page 58 of David Griffiths' *Introduction to Quantum Mechanics* (2nd edition, 2005).

Calculation establishes that if $\xi^2 = \frac{1}{m\omega}$ then

$$\tfrac{1}{2m}p^2(t) + \tfrac{1}{2}m\omega^2 x^2(t) = \tfrac{1}{2m}p_0^2 + \tfrac{1}{2}m\omega^2 x_0^2 = E$$

so the moving point $\{x(t), p(t)\}$ traces an isoenergetic ellipse in classical phase space, while

$$\begin{aligned}\psi_\alpha(x,t) &= \exp\left\{-\tfrac{1}{2}\tfrac{i}{\hbar}x(t)p(t)\right\}\\ &\qquad\cdot\left(\frac{1}{2\pi(\Delta x)^2}\right)^{\frac{1}{4}}\exp\left\{-\frac{1}{4}\left(\frac{x-x(t)}{\Delta x}\right)^2 + ip(t)x/\hbar\right\}\end{aligned} \tag{30.1}$$

$$\Downarrow$$

$$|\psi_\alpha(x,t)|^2 = \left(\frac{1}{2\pi(\Delta x)^2}\right)^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left(\frac{x-x(t)}{\Delta x}\right)^2\right\} \tag{30.2}$$

is a Gaussian which sloshes rigidly back and forth with period $\tau = 2\pi/\omega$ and

$$\text{amplitude} = \sqrt{2E/m\omega^2}$$

Equations (29) describe the motion of a "launched oscillator ground state."

Wigner representation of coherent states. With nothing more than a passing allusion to "work done some years ago by Leo Szilard and the present author for another purpose," E. P. Wigner[18] was content simply to pluck from his hat the construction

$$P_\psi(x,p) = \tfrac{2}{h}\int \bar{\psi}(x+\zeta)\,e^{2\frac{i}{\hbar}p\,\zeta}\,\psi(x-\zeta)\,d\zeta \tag{31}$$

that in the hands of J. H. Groenwald[19] and especially of J. E. Moyal[20] was to give rise to the "phase space formulation of quantum mechanics," to which a fairly comprehensive introduction can be found in Chapter 2 of my *Advanced Quantum Topics* (2000). It emerged that if the quantum observable A stands in "Weyl correspondence" with the classical observable

$$\begin{aligned}A(x,p) &= \iint a(\alpha,\beta)\,e^{\frac{i}{\hbar}(\alpha p+\beta x)}\,d\alpha d\beta\\ A(x,p) &\xrightarrow[\text{Weyl}]{} \mathsf{A} = \iint a(\alpha,\beta)\,e^{\frac{i}{\hbar}(\alpha\,\mathsf{p}+\beta\,\mathsf{x})}\,d\alpha d\beta\end{aligned}$$

and if $P(x,p)$ is the inverse Weyl transform of the density matrix $\boldsymbol{\rho}$, then

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[18] "On the quantum correction for thermodynamic equilibrium," Phys. Rev. **40**, 749 (1932).
[19] "On the principles of elementary quantum mechanics," Physica **12**, 405 (1946).
[20] "Quantum mechanics as a statistical theory," Proc. Camb. Phil. Soc. **45**, 92 (1949).

quantum expectation values assume formally the structure of the expectation values encountered in classical statistical mechanics:

$$\langle \mathbf{A} \rangle_\rho = \mathrm{tr}\,\mathbf{A}\,\boldsymbol{\rho} = \iint A(x,p)P(x,p)\,dxdp$$

In particular, one has

$$\mathrm{tr}\,\boldsymbol{\rho} = 1 = \iint P(x,p)\,dxdp$$

as one would expect if $P(x,p)$ referred to a "distribution on phase space." But —and here quantum theory asserts its exceptionalism—Wigner distributions can (and typically do) assume *negative values*. At no point does this circumstance compromise their quantum mechanical utitility, but in its light they are called "quasi-distributions."[21] It was known to Wigner, but first reported (without proof) by Takabayasi,[22] that $P(x,p)$ is bounded above and below by a universal constant that becomes infinite in the classical limit:

$$|P(x,p)| \leqslant \tfrac{2}{\hbar}$$

This in conjunction with $\int P(x,p)\,dxdp = 1$ means that $P(x,p)$ cannot have a "phase space footprint" with area less than $\frac{1}{2}\hbar$, which casts the Heisenberg uncertainty principle in interesting new light: a rectangle of that minimal area would have sides Δx and Δp that satisfy $\Delta x\,\Delta p = \frac{1}{2}\hbar$.

The phase space formulation of quantum mechanics is a lovely subject which, unfortunately, has been (at least until recently) largely ignored by the authors of quantum textbooks.[23] The subject has, however, acquired the status of an important tool in the hands particularly of physicists concerned with quantum optics and its application to foundational problems. It is, therefore, not surprising that the subject is touched upon in §3.7 of Garry & Knight.[5]

When the density operator refers to a pure state $\boldsymbol{\rho} = |\psi)(\psi|$ we recover precisely the Wigner distribution (31). If, in particular, we insert the coherent state (29) into (31) we (use *Mathematica*'s `Fourier Transform` command)

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[21] Feynman—among many others (including John Bell)—admitted to a fascination with the "negative probability" idea; for references, see pages 62-64 and 71-72 in Chapter 2 of *Advanced Quantum Topics* (2000).

[22] T. Takabayasi, "The formulation of quantum mechanics in terms of ensembles in phase space," Prog. Theo. Phys. **11**, 341 (1954). See especially §7 in that important paper.

[23] There is, for example, no mention of the subject in Griffiths or Mertzbacher, though Asher Peres, in Chapter 10 ("Semiclassical Methods") of his *Quantum Theory*: *Concepts and Methods* (1993) does in his §10-4 provide a brief (and not-very-helpful) account of its bare essentials.

obtain

$$P_\alpha(x,p) = \frac{2}{h}\exp\Big\{-\tfrac{1}{2}\Big(\frac{x-x_0}{\Delta x}\Big)^2 - \tfrac{1}{2}\Big(\frac{p-p_0}{\Delta p}\Big)^2\Big\} \tag{32}$$

with $\Delta p = \frac{1}{2}\hbar/\Delta x$. The associated marginal distributions are

$$\mathcal{X}_\alpha(x) = \int P_\alpha(x,p)\,dp = \frac{1}{\sqrt{2\pi}\,\Delta x}\exp\Big\{-\tfrac{1}{2}\Big(\frac{x-x_0}{\Delta x}\Big)^2\Big\} \tag{32.1}$$

$$\mathcal{P}_\alpha(p) = \int P_\alpha(x,p)\,dx = \frac{1}{\sqrt{2\pi}\,\Delta p}\exp\Big\{-\tfrac{1}{2}\Big(\frac{p-p_0}{\Delta p}\Big)^2\Big\} \tag{32.2}$$

Here as before, $\alpha = \frac{1}{\sqrt{2\hbar}}(\xi^{-1}x_0 + i\,\xi\,p_0)$ where x_0, p_0 and $\xi = \Delta x/\Delta p$ serve as control parameters. Note that the Wigner functions of coherent states are, according to (32)—quite uncharacteristically of Wigner functions in general—everywhere non-negative.

Set $x_0 = p_0 = 0$ and $\xi = 1/\sqrt{m\omega}\,$; then (32) becomes a description of the oscillator ground state. Displaced ground states (relax the conditions $x_0 = p_0 = 0$) are perfectly good oscillator states which $\mathbf{H}_{\text{osc}}$ sets into elliptical motion:

$$P_\alpha(x,p,t) = \frac{2}{h}\exp\Big\{-\tfrac{1}{2}\Big(\frac{x-x(t)}{\Delta x}\Big)^2 - \tfrac{1}{2}\Big(\frac{p-p(t)}{\Delta p}\Big)^2\Big\}$$

In x-representation the energy eigenstates of an oscillator (eigenstates $|n)$ of the number operator) can in terms of the variables

$$\begin{aligned} \varkappa &= \sqrt{2m\omega/\hbar}\cdot x \quad &&:\quad \text{dimensionless length}\\ \wp &= \sqrt{2/m\omega\hbar}\cdot p \quad &&:\quad \text{dimensionless momentum} \end{aligned}$$

be described

$$\begin{aligned} \psi_0(x) &= \tfrac{1}{\sqrt{a\sqrt{2\pi}}}e^{-\frac{1}{4}\varkappa^2}\\ \psi_1(x) &= \tfrac{1}{\sqrt{a\sqrt{2\pi}}}e^{-\frac{1}{4}\varkappa^2}\cdot\tfrac{1}{\sqrt{1!}}\varkappa\\ \psi_2(x) &= \tfrac{1}{\sqrt{a\sqrt{2\pi}}}e^{-\frac{1}{4}\varkappa^2}\cdot\tfrac{1}{\sqrt{2!}}(\varkappa^2-1) \end{aligned}$$

with $a = \sqrt{\hbar/2m\omega}$, and give rise to Wigner distributions that can be written

$$\left.\begin{aligned} P_0(x,p) &= +\tfrac{2}{h}e^{-\frac{1}{2}\mathcal{E}}\\ P_1(x,p) &= -\tfrac{2}{h}e^{-\frac{1}{2}\mathcal{E}}(1-\mathcal{E})\\ P_2(x,p) &= +\tfrac{2}{h}e^{-\frac{1}{2}\mathcal{E}}(1-2\mathcal{E}+\tfrac{1}{2}\mathcal{E}^2)\\ &\;\;\vdots\\ P_n(x,p) &= (-)^n\tfrac{2}{h}e^{-\frac{1}{2}\mathcal{E}}L_n(\mathcal{E}) \end{aligned}\right\} \tag{33}$$

Here $\mathcal{E} = \varkappa^2 + \wp^2$ is "dimensionless energy," interpretable as the squared radius of a circle inscribed on the $\{\varkappa,\wp\}$-coordinated phase plane, and the functions $L_n(\bullet)$ are Laguerre polynomials. At $n = 0$ we recover the Wigner function of the oscillator ground state. In all excited cases $(n > 0)$ the oscillator Wigner functions exhibit annular regions of negativity. In particular, $P_{\text{odd}}(0,0) < 0$.

In *Mathematica* 7 use variants of the following command

```
Plot3D[(-1)^3 Exp[-1/2(x^2+p^2)] LaguarreL[3,x^2+p^2],{x,-6,6},
{p,-6,6}, PlotRange→{-1,1}, PlotPoints→50,
Mesh→False, Boxed→False, Axes→False]
```

to see what such functions look like.

> At this point (4:27pm, 11 December 2012) I received word that Richard Crandall has been admitted to the ICU at OHSU suffering from acute leukemia. Chemotherapy is under way, is expected to continue for some time, and he reportedly "feels like shit."

In 1940, Kôdi Husimi (1909-2008)—then a young physicist at the University of Osaka—published a searching account of the theory of density operators.[24] Though somewhat tangential to Husimi's main interests (nuclear physics, plasma physics, statistical mechanics, origami. . . and later: the control of nuclear weapons), the paper demonstrated an exceptionally secure command of quantum developments during the 1930s, but—oddly—it appears that Husimi had never heard of Wigner distributions. The theory of which he proceeded to reinvent. . . and to extend: Husimi devised a pretty way to "temper" Wigner's quasi-distributions so that they become everywhere-non-negative proper distributions. Husimi's idea is easy to describe, though I must direct my reader elsewhere[25] for a detailed demonstration that it works as claimed: he introduces a "smear function" which is none other than the Wigner function (32) of a coherent state

$$G(x-x_0,p-p_0)=\tfrac{1}{\sigma\sqrt{2\pi}}\tfrac{1}{\lambda\sqrt{2\pi}}\exp\Big\{-\tfrac{1}{2}\big[\tfrac{x-x_0}{\sigma}\big]^2-\tfrac{1}{2}\big[\tfrac{p-p_0}{\lambda}\big]^2\Big\}$$

(where σ and λ are subject to the "minimality condition" $\sigma\lambda=\frac{1}{2}\hbar$) and proceeds

$$P_\psi(x,p)\xrightarrow[\text{Husimi}]{}\boldsymbol{P}_\psi(x,p)=h\iint G(x-x',p-p')P_\psi(x',p')\,dx'dp'$$

where a "poor man's bold" symbol has been used to suggest that $\boldsymbol{P}_\psi$ is a "smeared" companion of P_ψ. It can be shown without much difficulty that all Husimi functions are in fact everywhere-non-negative and normalized. But while Wigner functions satisfy

$$h\iint P_\psi^2(x,p)\,dxdp=1\quad:\quad\text{all }|\psi)$$

as an expression of the pure-state condition $\mathrm{tr}\,\boldsymbol{\rho}_\psi^2=1$, one finds that Husimi

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[24] "Some formal properties of the density matrix," Prog. Phys. Math. Soc. **22**, 264-314 (1940). The paper was published in English, but in a journal that was not widely available in the West until after the war; it took many years for its value to be appreciated.

[25] See once again *Advanced Quantum Topics* (2000), Chapter 2, pages 36-50.

smearing produces a result known to be characteristic of mixtures:

$$h\iint \boldsymbol{P}_\psi^2(x,p)\,dxdp < 1 \quad : \quad \text{all } |\psi)$$

The Husimi transforms of the oscillator energy eigen-distributions (33) are found to read

$$\left.\begin{aligned}
\boldsymbol{P}_0(x,p) &= \tfrac{1}{2}\tfrac{2}{h}e^{-\frac{1}{4}\mathcal{E}}\\
\boldsymbol{P}_1(x,p) &= \tfrac{1}{8}\tfrac{2}{h}e^{-\frac{1}{4}\mathcal{E}}\cdot\mathcal{E}\\
\boldsymbol{P}_2(x,p) &= \tfrac{1}{64}\tfrac{2}{h}e^{-\frac{1}{4}\mathcal{E}}\cdot\mathcal{E}^2\\
&\vdots\\
\boldsymbol{P}_n(x,p) &= \tfrac{1}{2\cdot 4^n\cdot n!}\tfrac{2}{h}e^{-\frac{1}{4}\mathcal{E}}\cdot\mathcal{E}^n
\end{aligned}\right\} \tag{34}$$

which are manifestly non-negative. One verifies that

$$\iint \boldsymbol{P}_n(x,p)\,dxdp = 1 \quad : \quad \text{all } n$$

but discovers that

$$\begin{aligned}
h\iint \boldsymbol{P}_0^2(x,p)\,dxdp &= \tfrac{1}{2} < 1\\
h\iint \boldsymbol{P}_1^2(x,p)\,dxdp &= \tfrac{1}{4} < 1\\
h\iint \boldsymbol{P}_2^2(x,p)\,dxdp &= \tfrac{3}{16} < 1\\
&\vdots
\end{aligned}$$

Historically (and still today), some people have looked upon the fact that Wigner distributions display regions of negativity as a fatal defect—reason enough to abandon the entire Wigner/Moyal formalism. Certainly those regions of negativity pose a problem if one proposes to use

$$\text{entropy} = -\iint P(x,p)\log P(x,p)\,dxdp$$

to assign an entropy to quantum states and mixtures, for it leads to a "complex entropy," a notion with which I have wrestled a bit elsewhere. Others have considered it to be evidence that the formalism stands in need of surgery. Husimi's procedure (and others have been proposed) can be viewed in this light.[26] But what I myself find most striking about Husimi's procedure is the novel use it makes of (the entire population of) coherent states.

[26] So little known did Husimi's work for so long remain that after a lapse of 36 years Nancy Cartwright—a philosopher of science then at Stanford—felt called upon to reinvent it: "A non-negative Wigner-type distribution," Physica **83A**, 210 (1976). Of course, neither Husimi nor Cartwright made mention of coherent states.

Yet another route to the coherent state concept. We looked first to the eigenstates of the operator a that arises when the oscillator Hamiltonian is factored, and were led to the overcomplete population of "coherent states" $|\alpha)$.[27] When we looked to the functions that minimize the expression on the right side of Schrödinger's inequality we (assuming that the observables in question are—like x and p—conjugate) were led back again to that same population of states, which acquired at this point the significance of "minimal uncertainty states," states that kill the quantum correlation coefficient. We encountered those states (actually the Wigner representations of those states) once again as the "Husimi smear functions" that remove regions of negativity from Wigner distributions. I turn now to review of yet another line of argument from which coherent states emerge as natural objects.[28]

In 1965, two relatively obscure Bell Labs engineers who were interested in "determining the limitations of coherent quantum mechanical amplifiers, etc." published a 5-page paper[29] that for nearly thirty years attracted very little attention.[30] It did, however, engage the immediate attention of C. Y. She and H. Heffner, two applied physicists at Stanford, who within seven months had submitted a paper that took Arthurs & Kelly as its point of departure.[31]

Arthurs & Kelly, elaborating on a dynamical measurement model proposed

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[27] Supersymmetric quantum mechanics exploits the analogous material that emerges *whenever* a positive semi-definite Hamiltonian can be written as a Wieshart product $\mathsf{H}_1 = \mathsf{W}^+\mathsf{W}$. In that theory the properties of H_1 are studied in conjunction with those of its companion $\mathsf{H}_2 = \mathsf{W}\mathsf{W}^+$. See Christopher Lee, "Supersymmetric quantum mechanics," (Reed College Thesis, 1999), which provides a good introductory review and an elaborate bibliography. See also pages 4-7 of my "Simultaneous measurement of noncommuting quantum observables," (October 2012). "Factorization methods" were pioneered by Dirac (1935) and Schrödinger (1940) and further developed in a famous paper by Leopold Infeld & T. E. Hull, "The factorization method," Rev. Mod. Phys. **23**, 21-68 (1951). A Google search reveals that the literature has assumed vast proportions. An interesting recent contribution is J. Oscar Rosas-Ortez, "On the factorization method in quantum mechanics," which is available on the web at arXiv:quant-ph/9812003v2 14 Oct 1999.

[28] It was this last line of argument that originally motivated this entire discussion.

[29] E. Arthurs & J. L. Kelly, Jr., "On the simultaneous measurement of a pair of conjugate observables," Bell System Technical Journal **44**, 725-629 (1965). The paper actually appeared in an appendage to that journal called BSTJ Briefs.

[30] The writing is densely obscure, and the authors claim a result that most physicists were prepared to find dubiously perplexing. Besides, it was not the habit of most physicists to scan the BSTJ, even though it was the journal in which Claude Shannon and his distinguished Bell colleagues usually published.

[31] C. Y. She & H. Heffner, "Simultaneous measurement of noncommuting observables," Phys. Rev. **152**, 1103-1110 (1966). Heffner maintained a working relationship with W. H. Louisell at Bell Labs.

by John von Neumann in the final pages of his *Mathematical Foundations of Quantum Mechanics* (1932), imagined a two-detector device that upon completion of its interaction with a system in initial state $\psi(x)$ announces $\{x_0, p_0\}$, signaling (as they show) that it has accomplished

$$\psi_{\text{before}}(x) \longrightarrow \psi_{\text{after}}(x) = \Big(\frac{1}{2\pi(\Delta x)^2}\Big)^{\frac{1}{4}} \exp\Big\{-\frac{1}{4}\Big(\frac{x-x_0}{\Delta x}\Big)^2 + ip_0 x/\hbar\Big\}$$

where the value of Δx is set by the specialized initial states of the detectors. The point of immediate relevance (see again (29)) is that as a byproduct of its activity the Arthurs/Kelly device prepares coherent states.

She & Heffner—who were familiar not only with quantum optics (then in its infancy) but also with the Bayesian information-theoretic approach to physics (and to science generally) propounded by E. T. Jaynes[32]—proposed to recover essential aspects of Arthurs/Kelly's result from an appeal to a *maximal entropy* principle. I turn a review of their argument, which they relegate to an appendix.

With She & Heffner, we seek the positive definite self-adjoint operator $\boldsymbol{\rho}$ that maximizes the von Neumann entropy

$$S = -\text{tr}\,(\boldsymbol{\rho}\log\boldsymbol{\rho})$$

subject to these five constraints

$$\begin{aligned}
\text{tr}\,\boldsymbol{\rho} &= 1 && (i)\\
\text{tr}\,(\mathbf{x}\,\boldsymbol{\rho}) &= x_0 && (ii)\\
\text{tr}\,(\mathbf{p}\,\boldsymbol{\rho}) &= p_0 && (iii)\\
\text{tr}\,(\mathbf{x}^2\boldsymbol{\rho}) &= x_0^2 + (\Delta x)^2 && (iv)\\
\text{tr}\,(\mathbf{p}^2\boldsymbol{\rho}) &= p_0^2 + (\Delta p)^2 && (v)
\end{aligned}$$

of which the first completes the list of requirements imposed upon density operators. We suppose x_0 and p_0 to be freely specifiable, but anticipate that Δx and Δp will ultimately become subject to the familiar constraint $\Delta x\,\Delta p \geqslant \frac{1}{2}\hbar$. Evidently $\boldsymbol{\rho}$ can be written

$$\boldsymbol{\rho} = \exp\big\{\lambda_0\,\mathbf{I} + \lambda_1\,\mathbf{x} + \lambda_2\,\mathbf{x}^2 + \lambda_3\,\mathbf{p} + \lambda_4\,\mathbf{p}^2\big\}$$

where the λs are Lagrange multipliers. To evaluate the traces we might bring

$$\big\{\lambda_0\,\mathbf{I} + \lambda_1\,\mathbf{x} + \lambda_2\,\mathbf{x}^2 + \lambda_3\,\mathbf{p} + \lambda_4\,\mathbf{p}^2\big\}\exp\big\{\lambda_0\,\mathbf{I} + \lambda_1\,\mathbf{x} + \lambda_2\,\mathbf{x}^2 + \lambda_3\,\mathbf{p} + \lambda_4\,\mathbf{p}^2\big\}$$

to $\mathbf{x}\,\mathbf{p}$-ordered form and make use of the "mixed representation trick."[33] But

[32] They cite E. T. Jaynes, "Information theory and statistical mechanics," Phys. Rev. **106**, 620-630 (1957), but more relevant is "Information theory and statistical mechanics. II," Phys. Rev. **108**, 171-190 (1957). Jaynes, by the way, had been a student of Wigner.

[33] See *Advanced Quantum Topics*, Chapter 0, page 38. A relevant ordering identity ("McCoy's theorem") appears on page 34. See also page 41.

that program is somewhat easier to execute if one expresses all $\{\mathsf{x},\mathsf{p}\}$-operators in terms of the $\{\mathsf{a},\mathsf{a}^+\}$-operators introduced at (23). Writing[34]

$$\begin{aligned}\log\boldsymbol{\rho} &= \mu_0 - \mu_1(\mathsf{a}^+ - \bar{a})(\mathsf{a} - a)\\ &= (\mu_0 + \bar{a}a - g^2\hbar) - \mu_1\big\{g^2\xi^{-2}\mathsf{x}^2 + g^2\xi^2\mathsf{p}^2\\ &\qquad\qquad - (a+\bar{a})g\xi^{-1}\mathsf{x} + i(a-\bar{a})g\xi\,\mathsf{p}\big\}\\ &= \mu_0 - \mu_1\mathsf{A}^+\mathsf{A}\quad\text{with}\quad \mathsf{A} = \mathsf{a} - a\mathsf{I}\end{aligned}$$

we could read off descriptions of $\{\lambda_0,\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, the role of which has been taken over by $\{\mu_0,\mu_1,\xi,\Re(a),\Im(a)\}$. We now have

$$\boldsymbol{\rho} = e^{\mu_0}\exp\big\{-\mu_1\mathsf{A}^+\mathsf{A}\big\}$$

which by $[\mathsf{A},\mathsf{A}^+] = \mathsf{I}$ and a corollary $e^{u\,\mathsf{AB}} = \exp\Big\{\dfrac{1-e^{-u\,\mathsf{C}}}{\mathsf{C}}\,\mathsf{A}\!:\!\mathsf{B}\Big\}$ of McCoy's theorem[33] assumes the ordered form

$$= e^{\mu_0}\exp\big\{-(1-e^{-\mu_1})\mathsf{A}^+\!:\mathsf{A}\big\}$$

where the colon signifies that all A^+s are to be placed left of all As, as in the following example: $e^{\mathsf{A}:\mathsf{B}} \equiv \sum\frac{1}{n!}\mathsf{A}^n\mathsf{B}^n$ (this notational convention is due to Schwinger). So we have

$$\begin{aligned}\operatorname{tr}\boldsymbol{\rho} &= \tfrac{1}{\pi}\iint(\alpha|\boldsymbol{\rho}|\alpha)\,d^2\alpha\\ &= e^{\mu_0}\tfrac{1}{\pi}\iint\exp\big\{(e^{-\mu_1}-1)(\bar{\alpha}-\bar{a})(\alpha-a)\big\}\,d^2\alpha\\ &= e^{\mu_0}(1-e^{-\mu_1})^{-1}\cdot\tfrac{1}{\pi}\iint e^{-\bar{\beta}\beta}\,d^2\beta\quad\text{with}\quad\beta = \sqrt{1-e^{-\mu_1}}\,(\alpha-a)\\ &= e^{\mu_0}(1-e^{-\mu_1})^{-1}\end{aligned}$$

which to achieve $\operatorname{tr}\boldsymbol{\rho} = 1$ enforces

$$e^{\mu_0} = 1 - e^{-\mu_1} \equiv \varepsilon \tag{35.1}$$

The density operator has at this point assumed the form the form

$$\boldsymbol{\rho} = \varepsilon\exp\big\{-\varepsilon\,\mathsf{A}^+\!:\mathsf{A}\big\}$$

Constraints (*ii*) and (*iii*) can be consolidated, to read

$$\operatorname{tr}(\mathsf{a}\,\boldsymbol{\rho}) = g(\xi^{-1}x_0 + i\xi p_0)$$

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[34] As a notational convenience I have dropped all I operators and adopted the abbreviation $g = \frac{1}{\sqrt{2\hbar}}$.

Arguing as before, we have

$$\begin{aligned}
\mathrm{tr}(\boldsymbol{\rho}\mathsf{a}) &= \tfrac{1}{\pi}\iint (\alpha|\boldsymbol{\rho}\mathsf{a}\,|\alpha)\,d^2\alpha\\
&= \tfrac{1}{\pi}\iint \alpha\cdot\varepsilon\,\exp\big\{-\varepsilon\,(\bar\alpha-\bar a)(\alpha-a)\big\}\,d^2\alpha\\
&= \tfrac{1}{\pi}\iint \Big(a+\varepsilon^{-\frac{1}{2}}\beta\Big)\cdot\varepsilon\,e^{-\bar\beta\beta}\,\tfrac{d^2\beta}{\varepsilon}\\
&= a
\end{aligned}$$

because $\frac{1}{\pi}\iint\exp(-|\beta|^2)\,d^2\beta = 1$ while $\iint\beta\exp(-|\beta|^2)\,d^2\beta$ vahishes by a parity argument. So we have

$$a = \tfrac{1}{\sqrt{2\hbar}}(\xi^{-1}x_0 + i\,\xi\,p_0) \tag{35.2}$$

It remains to work out the implications of (iv) and (v). To that end we observe that

$$\begin{aligned}
\mathsf{A}\ &\equiv \mathsf{a}\ \ - a = g\big(\xi^{-1}(\mathsf{x}-x_0) + i\,\xi\,(\mathsf{p}-p_0)\big)\\
\mathsf{A}^{+} &\equiv \mathsf{a}^{+} - \bar a = g\big(\xi^{-1}(\mathsf{x}-x_0) - i\,\xi\,(\mathsf{p}-p_0)\big)
\end{aligned}$$

give

$$\begin{aligned}
\mathsf{x}-x_0 &= \tfrac{1}{2}g^{-1}\xi\quad(\mathsf{A}^{+}+\mathsf{A})\\
\mathsf{p}-p_0 &= i\,\tfrac{1}{2}g^{-1}\xi^{-1}(\mathsf{A}^{+}-\mathsf{A})
\end{aligned}$$

so

$$\begin{aligned}
(\mathsf{x}-x_0)^2 &= \quad\tfrac{1}{4}g^{-2}\xi^2\quad(\mathsf{A}^{+}\mathsf{A}^{+} + 2\,\mathsf{A}\,\mathsf{A}^{+} + \mathsf{A}\,\mathsf{A} - \mathsf{I})\\
(\mathsf{p}-p_0)^2 &= -\tfrac{1}{4}g^{-2}\xi^{-2}(\mathsf{A}^{+}\mathsf{A}^{+} - 2\,\mathsf{A}\,\mathsf{A}^{+} + \mathsf{A}\,\mathsf{A} + \mathsf{I})
\end{aligned}$$

and therefore (use $\frac{1}{4}g^{-2} = \frac{1}{2}\hbar$ and exploit $\mathrm{tr}\,\mathsf{AB} = \mathrm{tr}\,\mathsf{BA}$)

$$\begin{aligned}
(\Delta x)^2 &= \mathrm{tr}\big[(\mathsf{x}-x_0)^2\boldsymbol{\rho}\big]\\
&= \tfrac{1}{2}\hbar\,\xi^2\,\mathrm{tr}\Big\{\mathsf{A}^{+}\mathsf{A}^{+}\boldsymbol{\rho} + 2\mathsf{A}^{+}\boldsymbol{\rho}\mathsf{A} + \boldsymbol{\rho}\mathsf{A}\,\mathsf{A} - \boldsymbol{\rho}\Big\}\\
&= \tfrac{1}{2}\hbar\,\xi^2\cdot\tfrac{1}{\pi}\iint(\alpha|\{\text{etc.}\}|\alpha)\,d^2\alpha\\
&= \tfrac{1}{2}\hbar\,\xi^2\cdot\tfrac{1}{\pi}\iint\Big\{\big[(\bar\alpha-\bar a)+(\alpha-a)\big]^2 - 1\Big\}\\
&\qquad\qquad\qquad\qquad\cdot\varepsilon\,\exp\big\{-\varepsilon\,(\bar\alpha-\bar a)(\alpha-a)\big\}d^2\alpha\\
&= \tfrac{1}{2}\hbar\,\xi^2\cdot\tfrac{1}{\pi}\iint\Big\{\varepsilon^{-1}(\bar\beta+\beta)^2 - 1\Big\}\,e^{-\bar\beta\beta}\,d^2\beta\quad:\quad\beta = x+iy\\
&\Downarrow\\
&= \tfrac{1}{2}\hbar\,\xi^2\cdot\big(\tfrac{2}{\varepsilon}-1\big)\\
&= \tfrac{1}{2}\hbar\,\xi^2\cdot\frac{1+e^{-\mu_1}}{1-e^{-\mu_1}}
\end{aligned}\tag{35.3}$$

Similarly

$$(\Delta p)^2 = \tfrac{1}{2}\hbar\,\xi^{-2}\cdot\frac{1+e^{-\mu_1}}{1-e^{-\mu_1}} \tag{35.4}$$

Equations (35.3/4) conjointly supply

$$\xi^2 = \frac{\Delta x}{\Delta p} \tag{35.5}$$

(which we had occasion to write already at (23) on page 12) and

$$\mu_1 = \log\left(\frac{(\Delta p)^2 + \frac{1}{2}\hbar\xi^{-2}}{(\Delta p)^2 - \frac{1}{2}\hbar\xi^{-2}}\right) = \log\left(\frac{\Delta x\Delta p + \frac{1}{2}\hbar}{\Delta x\Delta p - \frac{1}{2}\hbar}\right) \tag{35.6}$$

from which by (35.1) it follows that

$$\mu_0 = \log\left(1 - e^{-\mu_1}\right) \tag{35.7}$$

Equations (35.7/6/5/2) describe in terms of $\{x_0, p_0, \Delta x, \Delta p\}$ the values that must be ascribed to the parameters $\{\mu_0, \mu_1, \xi, \Re(a), \Im(a)\}$ if $\boldsymbol{\rho}$ is to conform to the constraints (i)-(v).

It is natural (and, as will emerge, instructive) to ask: Does

$$\boldsymbol{\rho} = \varepsilon \exp\left\{-\,\varepsilon\,\mathbf{A}^+\!:\mathbf{A}\right\} \tag{36}$$

describe a pure state ($\mathrm{tr}\,\boldsymbol{\rho}^2 = 1$) or a mixed state ($0 < \mathrm{tr}\,\boldsymbol{\rho}^2 < 1$)? Adjustment of the argument that led to (35.1) supplies

$$\begin{aligned}
\boldsymbol{\rho}^2 = e^{2\mu_0}\exp\left\{-\,2\mu_1\,\mathbf{A}^+\mathbf{A}\right\} &= e^{2\mu_0}\exp\left\{-\,(1-e^{-2\mu_1})\,\mathbf{A}^+\!:\mathbf{A}\right\}\\
&\Downarrow\\
\mathrm{tr}\,\boldsymbol{\rho}^2 &= e^{2\mu_0}(1-e^{-2\mu_1})^{-1}\\
&= \frac{1}{(1-e^{-\mu_1})^2(1-e^{-2\mu_1})} \equiv f(\mu_1)
\end{aligned}$$

Plot $f(\mu_1)$ and see that

$$\mathrm{tr}\,\boldsymbol{\rho}^2 \ \text{ is } \begin{cases} <0 & \text{if} \quad \mu_1 < 0\\ \infty & \text{if} \quad \mu_1 = 0\\ >1 & \text{if} \quad \mu_1 > 0\\ \to 1 & \text{as} \quad \mu_1 \to \infty \end{cases}$$

so the only tenable case is the last one: $\boldsymbol{\rho}$ becomes an admissible density operator if and only if $\mu_1 = \infty$, which by (35.6) requires $\Delta x\,\Delta p = \frac{1}{2}\hbar$, in which case $\boldsymbol{\rho}$ refers to a *pure* state.[35]

[35] Here She & Heffner dropped the ball: they assert without argument (citing only §VI of Glauber,[2] though Glauber does not really address the issue) that $\boldsymbol{\rho}$ is positive-definite if and only if $1 - e^{-\mu_1} \geqslant 0$, which requires $\mu_1 \geqslant 0$. But in those cases $\boldsymbol{\rho}$ *cannot* be positive-definite, since $\mathrm{tr}\,\boldsymbol{\rho}^2 > 1$ becomes consistent with $\mathrm{tr}\,\boldsymbol{\rho} = 1$ only if $\boldsymbol{\rho}$ possesses negative eigenvalues, as illustrated by the following simple example:

$$\mathbb{M} = \begin{pmatrix} 2 & 0\\ 0 & -1\end{pmatrix} \quad : \quad \mathrm{tr}\,\mathbb{M} = 1,\ \mathrm{tr}\,\mathbb{M}^2 = 3 > 1$$

If $\mu_1 = \infty$ then $\mu_0 = 0$, $\varepsilon = 1$ and (36) assumes the simple form

$$\begin{aligned}\boldsymbol{\rho} &= \exp\big\{-\,\mathsf{A}^+\!:\mathsf{A}\big\}\\ &= \exp\big\{-\,(\mathsf{a}^+ - \bar{a}):(\mathsf{a} - a)\big\}\end{aligned}\tag{37}$$

and the question arises: To *what* state does $\boldsymbol{\rho}$ projectively refer? She/Heffner proceed from the observation that (by (7.1))

$$\begin{aligned}(\alpha|\boldsymbol{\rho}|\beta) &= \exp\big\{-(\bar\alpha - \bar a)(\beta - a)\big\}\cdot(\alpha|\beta)\\ &= \exp\big\{-\bar\alpha\beta + \bar\alpha a + \beta\bar a - |a|^2\big\}\cdot\exp\big\{-\tfrac12|\alpha|^2 - \tfrac12|\beta|^2 + \bar\alpha\beta\big\}\\ &= \exp\big\{-\tfrac12|\alpha|^2 - \tfrac12|\beta|^2 - |a|^2 + \bar\alpha a + \beta\bar a\big\}\end{aligned}$$

while

$$\begin{aligned}(\alpha|a)(a|\beta) &= \exp\big\{-\tfrac12|\alpha|^2 - \tfrac12|a|^2 + \bar\alpha a\big\}\cdot\exp\big\{-\tfrac12|a|^2 - \tfrac12|\beta|^2 + \beta\bar a\big\}\\ &= \exp\big\{-\tfrac12|\alpha|^2 - \tfrac12|\beta|^2 - |a|^2 + \bar\alpha a + \beta\bar a\big\}\end{aligned}$$

So

$$\begin{aligned}&(\alpha|\boldsymbol{\rho}|\beta) = (\alpha|a)(a|\beta) \quad : \quad \text{all coherent states } |\alpha), |\beta)\\ &\qquad\Downarrow\\ &\boldsymbol{\rho} = |a)(a| \quad \text{where} \quad a = \tfrac{1}{\sqrt{2\hbar}}(\xi^{-1}x_0 + i\xi p_0) \qquad \text{by (35.2)}\\ &\qquad\qquad\qquad\qquad\qquad\quad \xi = \sqrt{\Delta x/\Delta p} \qquad \text{by (35.5)}\end{aligned}$$

Which is to say: The unique (pure) density operator that conforms to the constraints (i)-(v) projects onto the coherent state which, as we saw at (29), can in x-representation be described

$$\psi_a(x) = (x|a) = \Big(\frac{1}{2\pi(\Delta x)^2}\Big)^{\frac14}\exp\Big\{-\frac14\Big(\frac{x - x_0}{\Delta x}\Big)^2 + ip_0x/\hbar\Big\}$$

Alternatively, we might have proceeded

$$\begin{array}{ccc} P_a(x,p) & \xleftarrow[\text{inverse Weyl}]{\qquad\qquad} & \boldsymbol{\rho}\\ \Big\downarrow{\scriptstyle\text{Beck inversion trick}} & & \\ \psi_a(x) & & \end{array}$$

but I omit those details.[36]

She & Heffner drew inspiration from Jaynes' "maximal entropy principle," and it was the construction $S(\boldsymbol{\rho}) = -\mathrm{tr}(\boldsymbol{\rho}\log\boldsymbol{\rho})$ that supplied the preceding discussion with its analytical leverage. It is curious, therefore, that we have not had to construct an explicit description of $S(\boldsymbol{\rho})$—though we are in position to do so—and have not demonstrated that $S(\boldsymbol{\rho})$ is in fact *maximal*, though since $\boldsymbol{\rho}$ is unique (the element of a single-member set) it is necessarily so.

[36] See *Advanced Quantum Topics* (2000), Chapter 2, pages 13-14.

But why "coherent"? Possible generalizations. We found that the $\mathbf{a}$-algebra, which sprang historically from Dirac's approach to simple oscillatory theory, has a life quite independent of oscillators, and when pushed farther (Glauber, Bargmann) than its mechanical application demands gives rise to overcomplete eigenstates $|\alpha)$ that are called "coherent states" for reasons that no mechanical usage can justify. We found that those same states emerge when one looks for the states that reduce Schrödinger's inequality

$$(\Delta x)^2(\Delta p)^2 \geqslant \Big(\frac{[\mathbf{x},\mathbf{p}]}{2i}\Big)^2 + \Big[\text{quantum correlation coefficient}\Big]^2 \tag{38}$$

to equality, and that they might for that reason be called "minimal uncertainty states." They are, moreover, the states prepared by Arthurs/Kelly's idealized "simultaneous $\{x,p\}$-measurement device." And they are the states employed by Husimi to heal Wigner quasi-distributions of their regions of negativity. Finally, we found—in the She/Heffner work that inspired this essay (again using the resources of $\mathbf{a}$-algebra)—that an information-theoretic search for entropy-maximizing the density operator $\boldsymbol{\rho}_\text{after}$ that conforms to specified values of $\langle\mathbf{x}\rangle$, $\langle\mathbf{p}\rangle$, $\langle\mathbf{x}^2\rangle$ and $\langle\mathbf{p}^2\rangle$ leads back again to that same population of states, which might on that basis be called "maximal entropy states."

So why have the eigenstates $|\alpha)$ of $\mathbf{a}$-operators—for which our experience has suggested at least two plausible names—come to be called "coherent states"? A clue is provided by the observation that in the papers of Bargmann (1961/62) and Segal (1963) they do *not* bear that name, while a reference to "coherence" appears already in the *title* ("Coherent and incoherent states of the radiation field") of Glauber's paper (1963).[2] "Coherence" is a notion that arises in the physics of waves (originally optics and acoustics, later "wave mechanics"), as a precondition for the occurance of interference effects when two or more waves are superimposed ("it takes two to interfere," two to cohere). It is, therefore, a notion irrelevant to the physics of solitary waves, or the mechanics of systems with only one moving part (oscillators, for example, or particle-in-a-box systems). Quantum optics (bosonic field theory, which is effectively a theory of *systems* of oscillators) is, however, a subject into which correlation/coherence enter as essential *statistical* ideas. It is, I suppose, in the light of this conceptual linkage that the states which kill the quantum correlation term in (38) came to be called "coherent states."

The Schrödinger inequality is "binary" in that it posits the existence of two non-commuting operators ($\mathbf{A}$ and $\mathbf{B}$: "it takes two to correlate"), speaks statistically about a relationship between two experimental numbers: ΔA and ΔB. *By specialization* it speaks about cases in which $\mathbf{A}$ and $\mathbf{B}$ are *conjugate*: $[\mathbf{A},\mathbf{B}] = i\hbar\mathbf{I}$. It becomes natural to ask: Can the Arthurs/Kelly simultaneous measurement scheme and/or the She/Heffner argument be modified to accommodate cases in which the observables, though non-commuting, are *not* conjugate?[37] Noting that while conjugacy is a binary concept, non-commutivity

[37] Shu/Heffner appear to claim so in the title ("Simultaneous measurement of

is not, it becomes natural to ask: Can analogs of the Schrödinger inequality, or of the Arthurs/Kelly measurement scheme, or of the She/Heffner maximal entropy argument... be developed which contemplate the existence of *three or more* non-commutative observables $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots\}$? Is it possible, for example, to speak of an optimal simultaneous measurement of all three components of angular momentum (spin)? Finally, it would be of interest to describe (which we appear to be in position to do without much difficulty) the entropy change

$$\Delta S = S_{\text{post-measurement}} - S_{\text{pre-measurement}}$$

brought about by simultaneous measurement processes.[38]

noncommuting observables") of their paper, but do not really address the issue. Note that the $\mathbf{a}$-algebra, of which they make essential use, is itself based upon a conjugacy statement: $[\mathbf{a}, \mathbf{a}^{+}] = \mathbf{I}$.

[38] Aspects of this topic were addressed (inconclusively) in a Reed College physics seminar ("Quantum theory of measurements, and entropy") presented by F. J. Belinfante on 3 April 1985, of which I possess what is, I suspect, the only surviving copy of the unpublished text.